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1984 J. Phys. A: Math. Gen. 17 3053

(http://iopscience.iop.org/0305-4470/17/15/022)

J. Phys. A: Math. Gen. 17 (1984) 3053–3061. Printed in Great Britain

Emission of radiation by a strongly coupled plasma

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Received 4 April 1984, in final form 25 May 1984

Abstract. We present the theoretical radiation of a strongly coupled electron–proton plasma ($n_e = 10^{21}$ cm^{-3}, T between 55 eV and 200 eV). Quantum effects are introduced by calculating the Slater sum of the binary e–p system and deriving an effective potential from it. A classical treatment of the e–p collision is then possible and the Larmor formula gives the associated radiation. The power spectrum exhibits a low-frequency maximum up to 50% higher than the classical bremsstrahlung, and a more rapid decrease at high frequencies.

1. Introduction

The radiation of a dense plasma, even slightly non-ideal, is not well known, but its evaluation is needed to describe the energetic equilibrium of this medium. Such plasmas have been extensively studied but neglecting the effects of radiation. The state of the art has been fixed up by Deutsch *et al* (1981). We present here a model that includes quantum effects into an effective proton–electron potential. The classical treatment of the collision and of the radiation gives us an expression for the total radiated energy and for the power spectrum.

The plasma we are dealing with is a proton and electron one ($n_e = n_i = 10^{21}$ cm^{-3}) at a relatively high temperature (6×10^5K $< T < 2\times10^6$ K). Table 1 summarises the parameters of this plasma in the temperature range, the interparticle distance d, the Debye length λ_D and associated number of particles n_D in the Debye sphere, the de Broglie wavelength λ_B of the electron–proton system, the Landau length l_d and the ideality parameter $\Gamma = l_d/d$.

From the λ_B values, it appears that a quantum treatment is necessary. We introduce the Slater sum formalism, analogous to the classical Boltzmann factor in statistical mechanics (Landsberg 1971). In a first approximation we restrict ourselves to a binary short-range proton–electron interaction, so the effective potential ϕ derived from Slater sums has a mechanical meaning. The derivative $-\partial\phi/\partial r$ constitutes the real mechanical force related to the situation of an electron in this environment (Balescu 1976, p 245). This force will allow us to classically calculate the radiation of an electron colliding with a proton at a distance no larger than λ_B.

To solve the radiation problem, Valuev and Kurilenkov (1981) have studied a slightly different plasma ($n_e = 10^{18}$ to 10^{20} cm^{-3}, $T = 10^4$ K). The trajectory of the electron was a classical hyperbola around each proton leading to a Larmor radiation. The density effect was treated by a spectral absorption coefficient derived from the velocity autocorrelation function of the protons. The authors have noted that an

Table 1. Parameters of the electron–proton plasma for two typical temperatures.

T (K)	d (Å)	λ_D (Å)	n_D	λ_B (Å)	l_d(Å)	Γ
5×10^5	4.9	15.5	15.7	1.05	0.34	0.069
10^6	4.9	21.8	43.5	0.74	0.17	0.033

increasing disagreement occurs at higher densities because of a bad description of the electron trajectory. So we have built our model where all quantum effects are included in the derivation of the trajectory from the pseudo-potential. In § 5, we will discuss the validity of this assumption, especially at intermediate distances $r \geqslant \lambda_B$.

In § 2, we present the Slater sums method and their calculation using binary Coulomb functions. The effective potential is numerically deduced in order to describe the electron–proton collision classically. The Larmor formula gives an analytical form of the total radiated energy with the approximation of a straight line trajectory. A spectral analysis of this radiation is then performed and shows two main features by comparison with the classical bremsstrahlung model: a low-frequency higher maximum and a more rapid decrease at high frequency. As a conclusion we discuss the validity of our model and possible improvements by using high-order interactions, especially three-body collisions.

2. Calculation of the effective potential

In statistical mechanics, the probability density to find an N-particle system in the $\{\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_N\}$ configuration is given by

$$W_N(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_N) = Q_N^{-1} S_N(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_N)$$

where Q_N is the configuration integral

$$Q_N = (N!\lambda_B^{3N})^{-1} \int \exp(-\beta U_N)\, \mathrm{d}^3\mathbf{r}_1 \ldots \mathrm{d}^3\boldsymbol{r}_N.$$

U_N represents the interaction energy of the N-particles system, λ_B is the classical de Broglie wavelength and $S_N(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_N)$ is called the Slater sum of the system (Munster 1974, ch VI).

S_N is the quantum analogue of the classical Boltzmann factor $\exp(-\beta U)$ and can be written using the state vector $\psi_n(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_N)$

$$S_N(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_N) = N!\lambda_B^{3N} \sum_n |\psi_n(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_N)|^2 \exp(-\beta E_n).$$

Using only binary Slater sums, Ebeling *et al* (1968) and Minoo *et al* (1981) have extended this formalism to completely ionised plasmas of protons (p) and electrons (e).

In this frame, we define a proton–electron effective potential ϕ_{pe} by

$$S_{pe}(\boldsymbol{r}_1, \boldsymbol{r}_2) = (\lambda_B)_p^3(\lambda_B)_e^3 \sum_n |\Psi_n(\boldsymbol{r}_1, \boldsymbol{r}_2)|^2 \exp(-\beta E_n) = \exp(-\beta\phi_{pe}). \qquad (1)$$

Such an effective potential will be temperature dependent.

In the centre-of-mass coordinates, the binary Slater sum becomes

$$S_{\mathrm{pe}}(\boldsymbol{r})=8\pi^{3/2}\lambda_{\mathrm{pe}}^{3}\sum_{k}|\Psi_{k}(\boldsymbol{r})|^{2}\exp(-\beta E_{k}) \tag{2}$$

using μ, the reduced mass of the system, and $\lambda_{\mathrm{pe}}=\hbar[\beta/(2\mu)]^{1/2}$, the equivalent thermal wavelength. As the plasma is supposed completely ionised, we can neglect bound states in the radiation calculus and we use only the Coulomb wavefunctions of the continuum (Landau and Lifshitz 1970)

$$\Psi_{k}(\boldsymbol{r})=Y_{l}^{m}(\theta,\varphi)R_{k}(r), \qquad E_{k}=\hbar^{2}k^{2}/(2\mu)\geqslant 0.$$

After performing the partial waves expansion of the radial function $R_k(r)$ equation (2) becomes

$$S_{\mathrm{pe}}(r)=2\pi^{1/2}\lambda_{\mathrm{pe}}^{3}a_{0}^{-3}\sum_{l=0}^{\infty}(2l+1)\left[\int_{0}^{\infty}\frac{4ka_{0}\exp(-\lambda_{\mathrm{pe}}^{2}k^{2})}{1-\exp[-2\pi/(ka_{0})]}\left(\prod_{s=1}^{l}[s^{2}+(ka_{0})^{-2}]\right)\right.$$
$$\left.\times\frac{(2kr)^{2l}}{[(2l+1)!]^{2}}|F(l+1+\mathrm{i}/ka_{0},2l+2,2\mathrm{i}kr)|^{2}\,\mathrm{d}(ka_{0})\right].$$

To obtain a more tractable expression, we introduce the following set of variables (Ebeling *et al* 1968)

$$x=\lambda_{\mathrm{pe}}k, \qquad \xi=2\lambda_{\mathrm{pe}}/a_{0}=l_{d}/\lambda_{\mathrm{pe}}, \qquad \rho=r/\lambda_{\mathrm{pe}},$$

where a_0 is the Bohr radius, l_d the Landau length and ξ the interaction parameter which measures the ratio of the electrostatic potential energy to the kinetic one at a distance λ_{pe},

$$S_{\mathrm{pe}}(\rho\lambda_{\mathrm{pe}})=4\pi^{1/2}\xi\sum_{l=0}^{\infty}(2l+1)\left[\int_{0}^{\infty}\frac{x\exp(-x^{2})}{1-\exp(-\pi\xi/x)}\left(\prod_{s=1}^{l}[s^{2}+(\xi/2x)^{2}]\right)\right.$$
$$\left.\times\frac{(2x\rho)^{2l}}{[(2l+1)!]^{2}}|F(l+1+\mathrm{i}\xi/2x,2l+2,2\mathrm{i}x\rho)|^{2}\,\mathrm{d}x\right]. \tag{3}$$

The $F(\alpha,\gamma,z)$ function is the confluent hypergeometric series that converges for all finite z if γ is non-zero. According to (3), the binary proton–electron Slater sum can be approximated for $\rho<1$ by the expansion

$$S_{\mathrm{pe}}(\rho\lambda_{\mathrm{pe}})=\sum_{i=0}^{6}c_{i}\rho^{i}$$

wher the coefficients c_i are simply expressed using integrals $J_n(\xi)$. (See appendix 1 for detailed expressions of these quantities.) According to this expansion, we have performed a numerical calculation of the binary Slater sums S_{pe} at distances between 0 and λ_{pe} and in the range of temperature $55\ \mathrm{eV}<T<200\ \mathrm{eV}$. The lower limit for T corresponds to the highest possible value of the interaction parameter ($\xi=1$) and the higher limit ($\xi\simeq 0.5$) separates the region where the plasma is a kinetic one. Following equation (1), the effective potential $\phi_{\mathrm{pe}}=-k_{\mathrm{B}}T\ln(S_{\mathrm{pe}})$ was available as temperature-dependent sets of data.

In order to derive the radiation from ϕ_{pe}, we needed an analytical formula, fitting well with the data and quite easily tractable. These conditions were fulfilled by a four-parameter expansion deduced from the Coulomb potential

$$
\begin{aligned}
-\phi_{pe}(r) &= X_1 + X_2/(r+\Lambda) + X_3/(r+\Lambda)^2, && r \leq \lambda_{pe}, \\
\phi_{pe}(r) &= \phi_{pe}(\lambda_{pe}), && r > \lambda_{pe}.
\end{aligned}
\tag{4}
$$

The parameters X_1, X_2, X_3, Λ are temperature dependent and calculated by a least squares approximation from each set of potential data (20 to 50 points).

Table 2 contains these parametric values for typical temperatures of the plasma. We can observe that the effective potential is quite different from the Coulomb potential; mainly it takes a finite value at null distance. This feature is now currently used but here we must note that it occurs without special hypothesis, only as a consequence of the quantum terms in Slater sums. A characteristic length of ϕ_{pe} appears to be the spatial extent of the electron λ_{pe}. It measures the range of the potential and also constitutes a reference for the other typical lengths of the plasma. Comparison with the Landau length l_d separates the conditions where electrons are in ionic Coulomb fields ($\xi \gg 1$) and the present case ($\xi \leq 1$) without localisation. Outside the convergence disc ($r > \lambda_{pe}$), we assume that the whole environment contributes to building a constant potential. This is a first approximation but quite sufficient because we will see in § 3 that the trajectory is very slightly sensitive to the potential even where it is steep.

Table 2. Calculated parameters of the effective potential in the whole temperature range.

T (K)	ξ	λ_{pe} (au)	X_1 (au)	$-X_2$ (au)	X_3 (au)	Λ/λ_{pe}
6×10^5	1.03	0.51	1.32	4.73	1.40	0.73
7×10^5	0.95	0.47	1.49	5.48	1.68	0.76
8×10^5	0.89	0.44	1.63	6.17	1.95	0.78
9×10^5	0.84	0.42	1.77	6.82	2.20	0.80
10^6	0.80	0.40	1.89	7.43	2.44	0.82
1.5×10^6	0.65	0.33	2.40	10.10	3.48	0.87
2×10^6	0.56	0.28	2.76	12.16	4.38	0.90

3. Path and radiation of an electron

We have seen in § 1 the validity of calculations using an effective potential (which includes quantum effects) as a classical one, especially the interpretation of $\boldsymbol{F} = -\nabla\phi_{pe}$ as a classical mechanical force. The e–p interaction can be treated by solving the motion of an electron under the central force $\boldsymbol{F}$. The equation of motion (Goldstein 1950)

$$\mu\dot{\boldsymbol{v}} = [X_2/(r+\Lambda)^2 + 2X_3/(r+\Lambda)^3]\boldsymbol{r}/r \tag{5}$$

is numerical solved by means of an iterative Runge–Kutta method. Even for the lowest temperatures or for small impact parameter, the trajectory may be considered as a straight line. The deviation does not exceed 5% of the impact parameter b. Figure 1 recalls the collision geometry.

The total energy radiated during the e–p collision is classically given by the Larmor formula

$$W = \frac{2e^2}{3c^3} \times \frac{1}{4\pi\varepsilon_0} \int_{-\infty}^{+\infty} |\dot{\boldsymbol{v}}|^2\,\mathrm{d}t$$

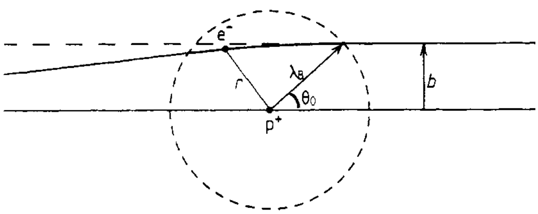

Figure 1. Scheme of the electron–proton collision.

in which $|\dot{v}|$ can be implicitly expressed following (5). The conservation of the angular momentum provides a relation $dt = r^2\,d\theta/(v_0 b)$; as the electron trajectory can be approximated to a straight line $b = r\sin\theta$ the total energy is simply given by

$$W = \frac{2e^2 b}{3c^3\mu^2 v_0} \times \frac{1}{4\pi\varepsilon_0}(I_4 X_2^2 + 4I_5 X_2 X_3 + 4I_6 X_3^2)$$

where

$$I_n = \int_0^{\pi} \frac{(\sin\theta)^{n-2}\,d\theta}{(b + \Lambda\sin\theta)^n}.$$

The I_4, I_5 and I_6 integrals are given in appendix 2.

We can note that the total energy is finite, but a global value hides much information about the radiation mechanism. So we will prefer to study the power spectrum.

4. Emission power spectrum

In order to calculate the spectral distribution of the emitted power, it is convenient to separate the electron acceleration into two components, a longitudinal one $\dot{v}_{\parallel} = \dot{v}\cos\theta$ along the trajectory, and a transverse one $\dot{v}_{\perp} = \dot{v}\sin\theta$. The angular variable θ (between 0 and π) may be introduced beneficially, using the relations seen previously in § 3. It is then possible to take $\tau = -\cot\theta = v_0 t/b$ as a reduced time and to write, putting $T_0 = (1+\tau^2)^{1/2}$,

$$\dot{v}_{\parallel} = -\frac{X_2\tau}{T_0(bT_0+\Lambda)^2} - \frac{2X_3\tau}{T_0(bT_0+\Lambda)^3}, \qquad \dot{v}_{\perp} = \frac{X_2}{T_0(bT_0+\Lambda)^2} + \frac{2X_3}{T_0(bT_0+\Lambda)^3}.$$

A numerical Fourier transform of these quantities is performed following the general formula

$$\varphi(\nu) = \int_{-\infty}^{+\infty} \dot{v}(\tau)\exp(-i2\pi\nu\tau)\,d\tau$$

and the power spectrum, as a function of the reduced frequency, becomes

$$P_\nu = \frac{2e^2}{3c^3}\frac{1}{4\pi\varepsilon_0}\frac{2b}{v_0}(\varphi_{\parallel}\varphi_{\parallel}^* + \varphi_{\perp}\varphi_{\perp}^*), \qquad \int_0^{+\infty} P_\nu\,d\nu = W.$$

Easier comparisons are possible with other results (Shkarofsky *et al* 1966) if we use the reduced pulsation $\omega = 2\pi\nu$, that leads simply to $P_\omega = P_\nu(2\pi)^{-1}$. Moreover, the potential parameters have been used in Hartree units (cf § 2) in order to work with numbers quite near unity; that introduces a unit P_0 which is temperature dependent via b and v_0:

$$P_0 = \frac{2e^2}{3c^3} \times \frac{1}{4\pi\varepsilon_0} \times \frac{b}{\pi v_0} \times \frac{e^2}{4\pi\varepsilon_0 a_0^2 \mu}.$$

Figure 2 shows that an increase of temperature causes an increase of the whole spectrum especially around the low-frequency maximum ($\omega b/v_0 \simeq 0.15$). At a given temperature (figure 3) a shorter impact angle displaces the maximum of the spectrum

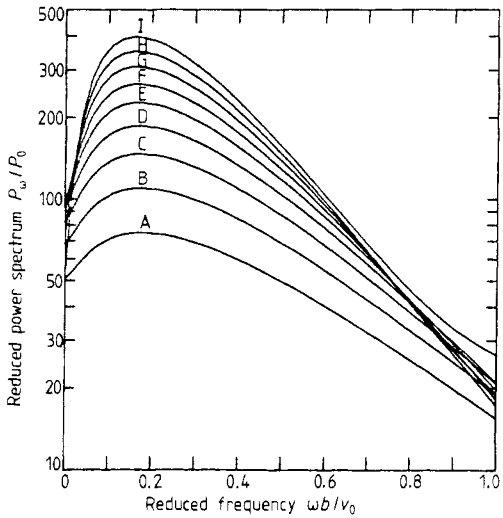

Figure 2. Calculated power spectra against reduced frequency for increasing temperatures from 6×10^5K to 1.4×10^6K and an impact angle $\theta_0 = 9\pi/20$. Curve A, 6×10^5K; B, 7×10^5K; C, 8×10^5K; D, 9×10^5K; E, 10^6K; F, 1.1×10^6K; G, 1.2×10^6K; H, 1.3×10^6K; I, 1.4×10^6K.

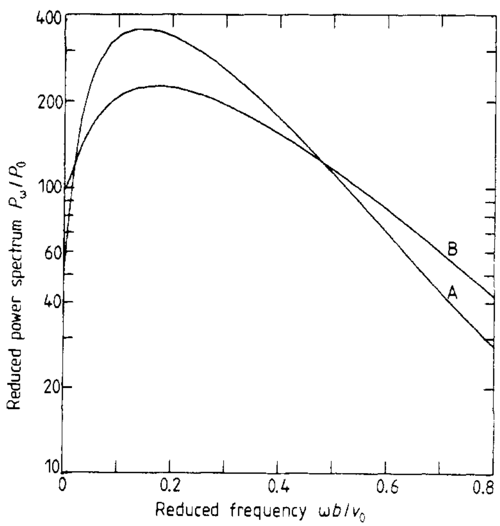

Figure 3. Effect of the impact angle θ_0 on the calculated power spectrum at $T = 10^6$K. Curve A: $\theta_0 = \pi/4$, curve B: $\theta_0 = 9\pi/20$.

and also increases its height. These results are consistent with the corresponding variations of the electron extent and of the collisional kinetics: the lower the temperature, the greater the size of the electron and the smoother the collision.

An interesting comparison is provided by figure 4 where we draw, in the same conditions of temperature and geometry (small angular deviation), the common bremsstrahlung radiation (Shkarofsky *et al* 1966) and our results. Shkarofsky *et al* have achieved an integration along a hyperbolic trajectory and obtained a flat spectrum. The low-frequency maximum is quite at the same place ($\omega b/v_0 \simeq 0.2$) and lower (-20 to -30%) but a dramatic difference appears at higher frequencies ($\omega b/v_0 > 0.5$) where our spectrum decreases very rapidly. As a consequence, we must note the integrability of our spectrum over all frequencies. A very simple comparison can be made at zero frequency where the classical bremsstrahlung is equal to $4(a_0/b)^4 P_0$.

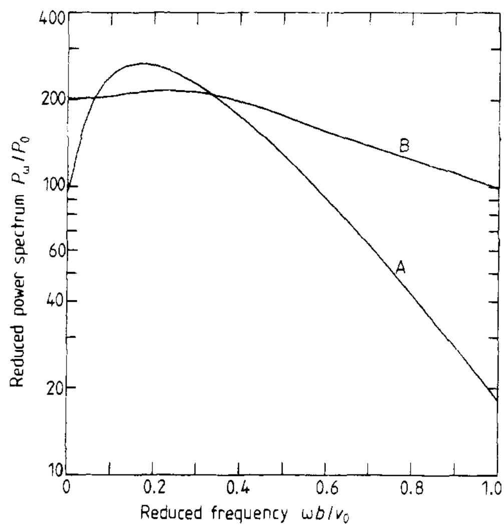

Figure 4. Comparison of calculated power spectra in the same conditions ($T = 1.1 \times 10^6$K, $\theta_0 = 9\pi/20$) using our model (A) and the classical bremsstrahlung (B).

5. Conclusion and prospects

We have shown the validity of using a pseudopotential to calculate the radiation emitted by a dense plasma with quantum effects. We can note that the previous results are quite similar to ours. As the former have been used up to now without major trouble, it is a confirmation of our calculations. But the first approximation of e–p binary short-range pseudopotential, including not only quantum effects but density effects, is still questionable.

We have neglected multipolar radiations coming from e–e collisions because their calculation is very intricate and it seems that their contribution is not significant. A real problem arises from intermediate distance configurations where no particular neighbour can be isolated. We have assumed a mean constant potential which is only the first approximation of the sum of all the contributions, continuously perturbed by the dynamics of the particles in the medium. This question is still open and deserves more attention. We must also take care of long-distance interactions where few electrons may intercalate between interacting particles: further results will be soon presented that take into account three-body interactions, mainly p–e–p and e–p–e.

Appendix 1. Power expansion coefficients for the Slater sum

$$J_n(\xi)=\int_0^{+\infty} x^n[1-\exp(-\pi\xi/x)]^{-1}\exp(-x^2)\,\mathrm{d}x,$$

$$c_0=4\pi^{1/2}\xi J_1(\xi),\qquad c_1=-\xi c_0,\qquad c_2=\tfrac{1}{2}\xi^2 c_0,$$

$$c_3=\pi^{1/2}[-5\xi^4 J_1(\xi)+4\xi^2 J_3(\xi)]/9,$$

$$c_4=\pi^{1/2}[7\xi^5 J_1(\xi)-20\xi^3 J_3(\xi)]/72,$$

$$c_5=\pi^{1/2}[-21\xi^6 J_1(\xi)+140\xi^4 J_3(\xi)-64\xi^2 J_5(\xi)]/1800,$$

$$c_6=\pi^{1/2}[11\xi^7 J_1(\xi)-140\xi^5 J_3(\xi)+224\xi^3 J_5(\xi)]/10800.$$

Appendix 2. Values of I_4, I_5 and I_6 integrals

Putting

$$\Delta=4(b^2-\Lambda^2),\qquad A=(-\Delta)^{-1/2}\ln\left(\frac{2\Lambda-(-\Delta)^{1/2}}{2\Lambda+(-\Delta)^{1/2}}\right),$$

the integrals used in the calculation of the total energy can be expressed as follows:

$$I_4=-\frac{32\Lambda}{3\Delta b^3}-\frac{64b\Lambda}{\Delta^2}\left(\frac{1}{2b^2}+\frac{3}{\Delta}\right)-96\frac{b-4\Lambda}{5\Delta^2}A,$$

$$I_5=\frac{128\Lambda^2}{81\Delta b^5}+\frac{16}{\Delta b^3}\left(\frac{4\Lambda^2}{7b^2}+\frac{1}{3}\right)+\frac{640b\Lambda^2}{3\Delta^3}\left(\frac{3}{2}+\frac{2\Lambda^2}{b^2}\right)\left(\frac{1}{2b^2}+\frac{3}{\Delta}\right)$$
$$+\frac{640b^2\Lambda^2}{\Delta^4}\left(\frac{3}{2}+\frac{2\Lambda^2}{b^2}\right)A,$$

$$I_6=-\frac{512\Lambda}{35\Delta b^5}\left(\frac{4\Lambda^2}{b^2}+1\right)-\frac{128\Lambda b^3}{15\Delta^3}\left(\frac{23}{b}+\frac{8\Lambda^2}{b^3}+\frac{2b}{\Delta}\right)\left[\frac{1}{b^3}+\frac{10b}{\Delta}\left(\frac{1}{2b^2}+\frac{3}{\Delta}\right)\right]$$
$$-\frac{1280b^3\Lambda^3}{5\Delta^5}\left(\frac{23}{b}+\frac{8\Lambda^2}{b^3}+\frac{2b}{\Delta}\right)A.$$

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